Solution to Assignment 11

Supplementary Problems

1. Let $\mathbf{a}, \mathbf{b} \in \mathbb{R}^n$. Establish Lagrange identity

$$\sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2 = |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \ .$$

Solution. We toy with indices.

$$\begin{aligned}\sum_{1\le i<j\le n} (a_i b_j - a_j b_i)^2 &= \sum_{1\le i\le j\le n} (a_i b_j - a_j b_i)^2 \\ &= \frac{1}{2}\sum_{1\le i,j\le n} (a_i b_j - a_j b_i)^2 \\ &= \sum_{1\le i,j\le n} a_j^2 b_i^2 - \sum_{1\le i,j\le n} a_i b_j a_j b_j \\ &= |\mathbf{a}|^2 |\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \ .\end{aligned}$$

An immediate consequence is the sharp Cauchy-Schwarz inequality:

$$|\mathbf{a}\cdot\mathbf{b}| \le |\mathbf{a}||\mathbf{b}| \ ,$$

and the equality sign holds if and only if the following two cases: (a) one of the vectors is the zero vector, or (b) $\mathbf{a}$ and $\mathbf{b}$ are proportional.

2. Deduce from (1) the identity

$$|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta \ ,$$

where θ is the angle between 3-vectors $\mathbf{a}$ and $\mathbf{b}$.

Solution. Taking $n = 3$ in (1), we have

$$\begin{aligned}|\mathbf{a}\times\mathbf{b}|^2 &= |\mathbf{a}|^2|\mathbf{b}|^2 - (\mathbf{a}\cdot\mathbf{b})^2 \\ &= |\mathbf{a}|^2|\mathbf{b}|^2(1-\cos^2\theta) \ , \quad \theta\in[0,\pi] \\ &= |\mathbf{a}|^2|\mathbf{b}|^2|\sin^2\theta| \ ,\end{aligned}$$

and the desired result comes from taking square root.

3. A regular parametrization $\mathbf{r}$ from the square $[0,1]^2$ to $S \subset \mathbb{R}^3$ is called a tube if (a) it is bijective on $(0,1)^2$ and (b) $\mathbf{r}((0,y)) = \mathbf{r}((1,y)), \ y \in [0,1]$. Show that for any irrotational C^1-vector field $\mathbf{F}$,

$$\oint_{C_1} \mathbf{F}\cdot d\mathbf{r} = \oint_{C_2} \mathbf{F}\cdot d\mathbf{r} \ ,$$

where $C_1 : x \mapsto \mathbf{r}(x,0)$ and $C_2 : x \mapsto \mathbf{r}(x,1)$ for $x \in [0,1]$.

Solution. Let C_1 be the curve $t \mapsto \mathbf{r}(t,0)$,$\gamma_1$ be $t \mapsto \mathbf{r}(1,t)$, C_2 be $t \mapsto \mathbf{r}(t,1)$, and γ_2 be $t \mapsto \mathbf{r}(0,t), \ t \in [0,1]$, so that the boundary of the surface is given by $C = C_1+\gamma_1-C_2-\gamma_2$. By Stokes' theorem and the assumption that $\nabla\times\mathbf{F} = \mathbf{0}$, we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S \nabla\times\mathbf{F}\cdot\mathbf{n}\, d\sigma = 0.$$

Since the line integrals over γ_1 and γ_2 cancel out.

4. (Optional) Let $\mathbf{r} : D \to S$ be a regular parametrization of S so that $\mathbf{r}_u \times \mathbf{r}_v$ is the chosen normal direction of S. Let $\gamma(t)$ be a parametrization of the boundary of D in anticlockwise direction. Show that the curve $\mathbf{r}(\gamma(t))$ described the boundary of S in the orientation induced by the chosen normal of S.

Solution. Let $\gamma(t), t \in [a, b]$, be a parametrization of the boundary of D in anticlockwise direction. Then $\mathbf{r}(\gamma(t))$ is a parametrization of the boundary of S. We want to show the direction of $\mathbf{r}(\gamma(t))$ is the same as the chosen orientation of S.

Recall that the orientation of S is described by $\mathbf{r}_u \times \mathbf{r}_v$ which should be a positive multiple of $\mathbf{n}$. The tangent of the boundary curve is a positive multiple of

$$\mathbf{w} \equiv \frac{d}{dt}\mathbf{r}(\gamma(t)) = \mathbf{r}_u u'(t) + \mathbf{r}_v v'(t) \ .$$

It suffices to show that $\mathbf{w} \times \mathbf{b}$ is a positive multiple of $\mathbf{r}_u \times \mathbf{r}_v$.

Let's move a horizontal line from $-\infty$ up until it hits the boundary of D at first time. Let $(x_0, y_0) = \gamma(t_0)$ be the touch point. At this point we have $\gamma'(t_0) = (a, 0), a > 0$. Then at the point $\mathbf{p} \equiv \mathbf{r}(\gamma(t_0))$, the tangent

$$\mathbf{t} = \alpha(\mathbf{r}_u u'(t_0) + \mathbf{r}_v v'(t_0)) = \alpha a \mathbf{r}_u,$$

for some $\alpha > 0$. Next, consider the line segment $t \mapsto (x_0, y_0 + t)$ which lies inside D for small $t \geq 0$. It follows that $\mathbf{r}(x_0, y_0 + t)$ lies on S. Therefore, $\dfrac{d}{dt}\mathbf{r}(x_0, y_0 + t) = \mathbf{r}_v(\mathbf{p})$ is a tangent vector at $\mathbf{p}$ pointing toward S. Hence it can be written as $\beta\mathbf{b} + \beta'\mathbf{t}$ for some $\beta > 0$. From $\mathbf{t} \times \mathbf{r}_v = \mathbf{t} \times (\beta\mathbf{b} + \beta'\mathbf{t}) = \beta\mathbf{t} \times \mathbf{b}$, we see that

$$\mathbf{t} \times \mathbf{b} = \beta^{-1}\mathbf{t} \times \mathbf{r}_v = \frac{1}{a\alpha\beta}\mathbf{r}_u \times \mathbf{r}_v \ ,$$

which is a positive multiple of $\mathbf{n}$.

We have shown that at the point $\mathbf{p}$ on the boundary, the orientation induced by γ coincides with the orientation induced by $\mathbf{r}$. By continuity, it must coincide at all other points.

Exercises 16.7

Using Stokes' Theorem to Find Line Integrals

In Exercises 1–6, use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

6. $\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + z\mathbf{k}$

   C: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above

Stokes' Theorem for Parametrized Surfaces

In Exercises 13–18, use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

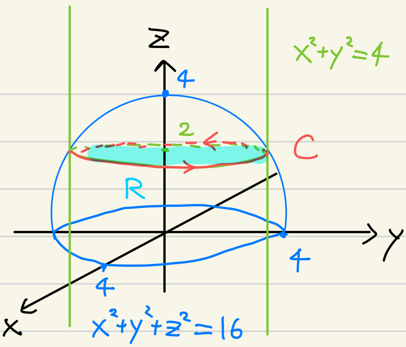
15. $\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$

    S: $\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}$,
    $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$

Theory and Examples

25. Find a vector field with twice-differentiable components whose curl is $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ or prove that no such field exists.

Q6 $\vec{F}(x,y,z) = x^2y^3\,\vec{i} + \vec{j} + z\vec{k}$.



$$\text{curl}\,\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & z \end{vmatrix} = -3x^2y^2\,\vec{k}\ ;$$

Let $f(x,y,z) = x^2+y^2+z^2$;

$$\nabla f(x,y,z) = 2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}\ ;\ \nabla f \cdot \vec{k} = 2z.$$

$$\therefore\ \vec{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\,\vec{i} + 2y\,\vec{j} + 2z\,\vec{k}}{2\sqrt{x^2+y^2+z^2}} = \frac{1}{4}\left(x\,\vec{i} + y\,\vec{j} + z\,\vec{k}\right).$$

$$d\sigma = \frac{|\nabla f|}{|\nabla f \cdot \vec{k}|}\,dA = \frac{8}{2z}\,dA = \frac{4}{z}\,dA.$$

$$\text{curl}\,\vec{F} \cdot \vec{n} = (-3x^2y^2) \cdot \frac{z}{4} = -\frac{3}{4}x^2y^2z.$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \iint_R \left(-\frac{3}{4}x^2y^2z\right)\left(\frac{4}{z}\right) dA = -3\iint_R x^2y^2\,dA$$

$$= -3\int_0^{2\pi}\int_0^2 (r\cos\theta)^2 (r\sin\theta)^2\, r\,dr\,d\theta$$

$$= -3\left(\int_0^{2\pi} \cos^2\theta \sin^2\theta\,d\theta\right)\left(\int_0^2 r^5\,dr\right)$$

$$= -\frac{3}{4}\int_0^{2\pi} \sin^2 2\theta \cdot \left[\frac{r^6}{6}\right]_0^2$$

$$= -\frac{3}{4}\int_0^{2\pi}\left(\frac{1-\cos 4\theta}{2}\right)d\theta \cdot \frac{32}{3} = -4\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{2\pi} = -8\pi$$

Q15 $\vec{F}(x,y,z) = x^2y\,\vec{i} + 2y^3z\,\vec{j} + 3z\,\vec{k}.$

$$\operatorname{curl}\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & 2y^3z & 3z \end{vmatrix} = (0-(2y^3))\,\vec{i} + (0-(x^2))\,\vec{k} = -2y^3\vec{i} - x^2\vec{k}.$$

$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r\,\vec{k}$, where $0 \le r \le 1$ and $0 \le \theta < 2\pi$.

$\vec{r}_r(r,\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \vec{k}$; $\vec{r}_\theta(r,\theta) = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (0-r\cos\theta)\,\vec{i} - (0+r\sin\theta)\,\vec{j} + (r\cos^2\theta + r\sin^2\theta)\,\vec{k}$$

$$= -r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\,\vec{k}.$$

$$\operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \operatorname{curl}\vec{F}\cdot(\vec{r}_r \times \vec{r}_\theta)\,dr\,d\theta$$

$$= \left(-2(r\sin\theta)^3\,\vec{i} - (r\cos\theta)^2\,\vec{k}\right)\cdot\left(-r\cos\theta\,\vec{i} - r\sin\theta\,\vec{j} + r\,\vec{k}\right)dr\,d\theta$$

$$= \left(2r^4\sin^3\theta\cos\theta - r^3\cos^2\theta\right)dr\,d\theta$$

$$\therefore \iint_S \operatorname{curl}\vec{F}\cdot\vec{n}\,d\sigma = \int_0^{2\pi}\!\!\int_0^1 \left(2r^4\sin^3\theta\cos\theta - r^3\cos^2\theta\right)dr\,d\theta$$

$$= \int_0^{2\pi}\left[\frac{2r^5}{5}\sin^3\theta\cos\theta - \frac{r^4}{4}\cdot\cos^2\theta\right]_0^1 d\theta$$

$$= \int_0^{2\pi}\left(\frac{2}{5}\sin^3\theta\cos\theta - \frac{1}{4}\cdot\cos^2\theta\right)d\theta$$

$$= \int_0^{2\pi}\left(\frac{2}{5}\sin^3\theta\cos\theta - \frac{1}{4}\cdot\left(\frac{1+\cos 2\theta}{2}\right)\right)d\theta$$

$$= \left[\frac{2}{5}\cdot\frac{\sin^4\theta}{4} - \frac{1}{8}\theta + \frac{\sin 2\theta}{16}\right]_0^{2\pi} = -\frac{\pi}{4}\;//$$

Q25 Prove by contradiction: Suppose on the contrary,

there exists $\vec{F}(x,y,z) = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k}$

such that $\text{curl}\,\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$, then

$$\text{div}(\text{curl}\,\vec{F}) = \frac{\partial}{\partial x}(R_y - Q_z) + \frac{\partial}{\partial y}(-R_x + P_z) + \frac{\partial}{\partial z}(Q_x - P_y) = 0$$

On the other hand, $\text{div}(x\vec{i} + y\vec{j} + z\vec{k}) = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z) = 1+1+1 = 3.$

This is a contradiction. Therefore, there does not exist $\vec{F}(x,y,z)$

such that $\text{curl}\,\vec{F} = x\vec{i} + y\vec{j} + z\vec{k}$.